Math 246B Lecture 8 Notes

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1 The Phragmén-Lindelöf Principle

1.1 The Phragmén-Lindelöf Principle for subharmonic functions

To prove the Phragmén-Lindelöf[1] principle, let's introduce some notation.

Definition 1.1. Let $\Omega \subseteq \mathbb{R}$ be open and unbounded. We say that $\varphi : \overline{\Omega} \to \mathbb{R}$ is a **Phragmén-Lindelöf function** for Ω if

1. $\varphi(x) > 0$ for large $|x|$.
2. If u is upper semicontinuous on $\overline{\Omega}$, subharmonic in Ω, $u \leq M$ on $\partial\Omega$, and $u(x) \leq \varphi(x)$ for large $x \in \overline{\Omega}$, then $u \leq M$ on $\overline{\Omega}$.

Remark 1.1. Let φ be a PL function for Ω. Let $f \in \mathrm{Hol}(\Omega) \cap C(\overline{\Omega})$ be such that $|f| \leq M$ on $\partial\Omega$ and $|f(z)| \leq e^{\varphi(z)}$ for large $z \in \overline{\Omega}$. Then $|f| \leq M$ on $\overline{\Omega}$.

Given Ω, how do we construct PL functions for Ω?

Theorem 1.1 (Phragmén-Lindelöf principle)**.** *Let $\Omega \subseteq \mathbb{R}^2$ be open and unbounded. Let $\psi : \overline{\Omega} \to [0, \infty)$ be such that*

1. *ψ is lower semicontinuous on Ω ($-\psi$ is upper semicontinuous),*
2. *ψ is super harmonic in Ω ($-\psi$ is subharmonic),*
3. *$\psi(x) \to +\infty$ as $|x| \to \infty$ for $x \in \overline{\Omega}$.*

Let $\varphi > 0$ be such that $\varphi(x) = o(\psi(x))$ when $|x| \to \infty$ for $x \in \overline{\Omega}$. Then φ is a PL function for Ω.

Here is the original argument by Phragmén and Lindelöf.

[1] Lindelöf was the teacher of Ahlfors.

Proof. Let u be upper semicontinuous on $\overline{\Omega}$, subharmonic in Ω, $u \leq M$ on $\partial\Omega$, and $u(x) \leq \varphi(x)$ for large $x \in \overline{\Omega}$. We want to show that $u \leq M$ on $\overline{\Omega}$. For $\varepsilon > 0$, set $u_\varepsilon = u - \varepsilon\psi$. Then u_ε is upper semicontinuous on $\overline{\Omega}$, subharmonic in Ω, $u_\varepsilon \leq M$ on $\partial\Omega$, and for large $x \in \overline{\Omega}$,

$$u_\varepsilon(x) \leq \varphi(x) - \varepsilon\psi(x) = -\psi(x)\left(\varepsilon - \frac{\varphi(x)}{\psi(x)}\right) \xrightarrow{|x|\to\infty} -\infty.$$

Let $a \in \Omega$, and let $R > |a|$ be such that $u_\varepsilon(x) \leq M$ for $|x| = R$ and $x \in \overline{\Omega}$. If $\Omega_R = \{x \in \Omega : |x| < R\}$, then $\partial\Omega \subseteq \partial\Omega \cup \{x \in \overline{\Omega} : |x| = R\}$, and $u_\varepsilon \leq M$ on $\partial\Omega_R$. Apply the maximum principle to u_ε and the bounded domain Ω_R to get that $u_\varepsilon \leq M$ on Ω_R. So

$$u_\varepsilon(a) = u(a) - \varepsilon\psi(a) \leq M.$$

Letting $\varepsilon \to 0^+$, we get that $u \leq M$ on Ω. So φ is a PL function for Ω. □

1.2 Phragmén-Lindelöf for a sector

This important case of the theorem was the original motivation for Phragmén and Lindelöf.

Theorem 1.2 (PL for a sector). *Let* $\Omega = \{z \in \mathbb{C}\setminus\{0\} : \alpha < \arg(z) < \beta\}$ *for* $0 < \beta - \alpha < 2\pi$. *Then* $\varphi(z) = |z|^k$ *is a PL function for* Ω *if* $0 < k < \pi/(\beta - \alpha)$.

Proof. We may assume after a rotation that $\Omega = \{z \in \mathbb{C} \setminus \{0\} : |\arg(z)| < \gamma/2\}$, where $0 < \gamma = \beta - \alpha < 2\pi$. Let $k < k_1 < \pi/\gamma$, and consider $\psi(z) = \mathrm{Re}(z^{k_1}) = \mathrm{Re}(e^{k_1 \log(z)})$, using the principal branch of log. This is $\psi(z) = |z|^{k_1}\cos(k_1 \arg(z))$ for $z \in \overline{\Omega}$ with $z \neq 0$. Then ψ is harmonic in Ω, continuous in $\overline{\Omega}$, and $|\psi(z)| \sim |z|^{k_1}$ since $|k_1 \arg(z)| \leq k_1\gamma/2 < \pi/2$. In particular, $\phi = o(\psi)$ at ∞. Therefore, φ is a PL function for Ω. □

Corollary 1.1 (classical PL principle). *Let* $\Omega = \{z \in \mathbb{C} \setminus \{0\} : \alpha < \arg(z) < \beta\}$, *where* $0 < \beta - \alpha < 2\pi$. *Let* $f \in \mathrm{Hol}(\Omega) \cap C(\overline{\Omega})$, *where* $|f| \leq M$ *on* $\partial\Omega$. *Assume that* $|f(z)| \leq C_1 e^{C_2|z|^k}$ *as* $|z| \to \infty$ *for* $z \in \overline{\Omega}$, *where* $0 < k < \pi/(\beta - \alpha)$. *Then* $|f| \leq M$ *on* $\overline{\Omega}$.

Here is an example from the spring 2015 analysis qualifying exam.

Example 1.1. Let $f \in \mathrm{Hol}(\mathbb{C})$ be such that $|f(z)| \leq e^{|z|}$ and $\sup_{x\in\mathbb{R}}(|f(x)|^2 + |f(ix)|^2) < \infty$. Show that f is constant.

Apply the classical Phragmén-Lindelöf principle 4 times, once to each quadrant. Then f is bounded, so f is constant by Liouville's theorem.

1.3 Phragmén-Lindelöf for general domains

Let $\Omega, \tilde{\Omega} \subseteq \mathbb{C}$ be open and unbounded, and let $G : \Omega \to \tilde{\Omega}$ is an an analytic isomorphism such that G extends to a homeomorphism $\overline{\Omega} \to \overline{\tilde{\Omega}}$. Then $|G(z)|$ is large iff $|z|$ is large.Then if φ is a PL function for $\tilde{\Omega}$, $\varphi \circ G$ is a PL function for Ω. (To check this, use that if $u \in SH(\tilde{\Omega})$, then $u \circ G \in SH(\Omega)$.)

Proposition 1.1. *Let* $\Omega = \{z \in \mathbb{C} : \mathrm{Im}(z) > 0, \alpha < \mathrm{Re}(z) < \beta\}$*. Then* $\varphi(z) = e^{k\,\mathrm{Im}(z)}$ *is a PL function for* Ω *for any* $0 < k < \pi/(\beta - \alpha)$*.*

We will prove this next time. The idea is that we find a conformal map from the half-strip to a sector with a disc removed. The map is $f(z) = e^{-icz}$ for some $0 < c < 2\pi/(\beta - \alpha)$.